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Surface adsorption of a self-avoiding polymer chain on a family of finitely ramified fractals

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Abstract. We study the surface adsorption of a flexible self-avoiding polymer chain on a family of Sierpinski gasket-type fractals using the real space renormalization group technique. The members of this family are characterized by an integer scale factor b which runs from 2 to ∞ . The Hausdorff dimension of these fractals tends to 2 from below as $b \rightarrow \infty$. We calculate the crossover exponent ϕ for the desorption transition exactly for $b=2$ to 6. For $b \rightarrow \infty$, we use finite size scaling arguments to show that $\phi \rightarrow 1/4$.

1. Introduction

Critical phenomena at surfaces have been extensively studied in recent years in the framework of equilibrium phase transition [1–6]. These studies include several important physical problems such as wetting, surface reconstruction and polymer adsorption [7–8]. Here we are concerned with the phenomena of polymer adsorption on a rigid, impenetrable substrate. This problem has been the focus of much attention in recent years because of its technological importance in the stabilization of colloidal dispersions used in paints, pharmaceuticals and foodstuffs, in lubrication, adhesion and membrane-phenomena and in the development of artificial organs. While in practical applications one does not usually operate in the vicinity of the critical point, it is very desirable to understand how changes in the attractive interaction strength at the surface affect the adsorption behaviour of the polymer for all values of the interaction strength.

In most of the theoretical work reported in the literature, consideration is given to adsorption of an isolated long polymer chain in good solvent on an attractive wall [5–8]. For this case there is an unbinding temperature analogous to a tricritical point, and in its vicinity a crossover regime is observed, where a simple scaling law holds [4]. There is a formal equivalence between the problem of polymer adsorption on a surface and the problem of critical phenomena in the n -vector model of a magnet with a free surface which has been used in scaling analysis [9]. Both the surface and bulk critical exponents have been calculated using renormalization group methods [5, 10, 11], exact enumeration methods and Monte Carlo simulations [12, 6]. For a two-dimensional system exact values of the exponents have been found by using conformal invariance [13, 14].

For a space of fractal dimension d_F and an adsorbing surface of dimension d_s , Bouchaud and Vannimenus [11] (hereinafter referred to as BV) have shown using scaling analysis that the crossover exponent ϕ has lower and upper bounds given as

$$1 - (d_F - d_s)\nu \leq \phi \leq d_s/d_F. \quad (1.1)$$

Here ν is the bulk radius of gyration exponent for a polymer chain in a good solvent and the exponent ϕ governs the region where critical scaling holds. At the critical point the number of monomers in direct contact with the surface varies as N^ϕ where N is the total number of monomers. For the proximal exponent m , which governs the variation of monomer density in the neighbourhood of the surface in the critical region, they found

$$m = d_F - d_s + (\phi - 1)/\nu. \quad (1.2)$$

Since for a regular lattice $d_s = d_F - 1$, equation (1.2) reduces to an expression conjectured by de Gennes and Pincus [15].

Many attempts have been made in recent years to calculate the values of ϕ and m on Euclidean lattices using real space renormalization methods [10, 11, 15]. However, as a rule, these models are not solvable analytically and numerical methods are often quite insufficient in the study of critical behaviours. The value of ϕ can, however, be calculated exactly for finitely ramified fractals using the RSRG method. This has been shown recently by BV for Sierpinski gaskets having rescaling parameter $b=2$ and embedded in two and three dimensions. An adsorption transition was found to take place when one of the boundaries of the fractal lattices was made attractive. Note that for a 3D Sierpinski gasket the adsorbing surface itself is a 2D gasket with $d_s = \ln 3/\ln 2$. Modelling the polymer on this surface by a trail silhouette, Orlandini *et al* [16] have recently shown the existence of two transitions in the adsorbed region, the first one, from linear to branched polymer behaviour, is followed by a further collapse into compact globule.

In this article we consider a family of Sierpinski-type fractals (or equivalently a Given-Mandelbrot family of fractals [17]) embedded in two dimensions and characterized by the rescaling parameter b which runs from 2 to ∞ . The fractal dimension d_F of members of this family is easily seen to be

$$d_F = \frac{\ln \left[b \left(\frac{b+1}{2} \right) \right]}{\ln b}. \quad (1.3)$$

As b is increased both the fractal and the spectral \bar{d} dimensions of the fractal increase monotonically and tend towards 2 from below. However, dimension d_s of the adsorbing surface remains one in all cases.

The bulk critical exponents of self-avoiding walks (SAWs) which simulate a long polymer chain in a good solvent have been calculated by Elezovic *et al* [18] for the family of Sierpinski type fractals with $2 \leq b \leq 8$. In a previous paper [19] the Given-Mandelbrot family of fractals was used to show that the bulk critical exponents of SAWs for large b can be expressed in a systematic expansion. These two families of fractals have same recursive construction, and hence all the critical properties. In this

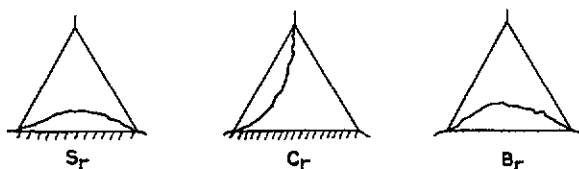


Figure 1. The two surface partition functions (S and C) and the bulk generating function (B) for the polymers on the Sierpinski gasket.

paper we use the finite size scaling theory to determine the behaviour of the crossover exponent ϕ for large b .

Study of the critical behaviour of a polymer chain interacting with a surface on these fractal lattices is of importance for several reasons: first, one can find exact values of ϕ as b is varied. By finding out how the crossover exponent changes on systematically changing the properties of the fractal, one can understand better how geometry affects tricritical behaviour. Further, one is often interested in the study of polymers in the presence of disorder in the surrounding medium. In many cases the disordered medium may be modelled as a random fractal, and the study of polymer adsorption on regular fractals is a prerequisite for such studies. The behaviour of ϕ as $b \rightarrow \infty$ can also be checked to see if it tends to its two-dimensional value $\frac{1}{2}$.

The paper is organized as follows. In section 2 we describe the real space renormalization group method (RSRG) to calculate the crossover exponent for self-avoiding walks interacting with a surface of a fractal lattice. We show that a model containing three parameters is adequate to describe the adsorbed phase and the unbinding transition of the SAWs. Section 3 is devoted to finite size scaling theory to calculate the values of the crossover exponent as a function of b , as $b \rightarrow \infty$. The paper ends with a brief discussion given in section 4.

2. Real space renormalization group calculation of crossover exponent ϕ

To perform an RSRG calculation on a polymer chain interacting with a surface, we study how the coupling constants describing a SAW change upon repeated length rescaling of the system. When these quantities remain invariant, the chain is 'self-similar' on all length scales and this is a 'fixed-point' of the rescaling transformation. In principle, an infinite number of parameters are required to describe exactly the chain and its behaviour under rescaling. It is clearly not possible to deal with this infinite parameter space and somewhere there must be truncation. For these fractals the bulk critical exponents are calculated using only one parameter which represents fugacity per monomer of the polymer chain [18–22]. Here, following *bv*, we introduce two more parameters denoting interaction of a monomer in the surface layer and in the adjacent one [11].

A walk is called a surface walk (and assigned configuration S) when it enters through one corner of the surface and leaves from the other as shown in figure 1. A bulk walk represented by B has no step on the surface or on the bonds connecting the surface with the bulk. A walk which enters through one corner of the surface and ends

up in the bulk is assigned configuration C . The generating functions for these walks can be written as

$$B_r(x) = \sum B_r(N)x^N \quad (2.1)$$

$$S_r(x, \omega, t) = \sum S_r(N, M, R)x^N \omega^M t^R \quad (2.2)$$

and

$$C_r(x, \omega, t) = \sum C_r(N, M, R)x^N \omega^M t^R. \quad (2.3)$$

As has already been indicated, x is a weight factor (fugacity) associated with each visited site of the lattice. $B_r(n)$ is the number of distinct configuration of SAWs which join two vertices of the r th order gasket in the bulk and N is the number of sites visited by the SAW. Here M and R represent, respectively, the number of visited sites of the lattice which lie on the surface and on a layer adjacent to the surface. The summations in (2.1)–(2.3) are over repeated indices and $S_r(N, M, R)$ and $C_r(N, M, R)$ represent number of configurations of respective walks of the r th order gasket. Note that apart from the fugacity x , the other two parameters $\omega = \exp(-E_s/T)$ and $t = \exp(-E_c/T)$ chosen here correspond to surface energy E_s denoting the interaction of a monomer (which is supposed to lie on a lattice site) with the surface, and E_c the interaction of a monomer with the adjacent layer.

For a 2D Sierpinski gasket with $b = 2$, the restricted partition functions are shown in figure 1. For this case the algebraic recursion relations are simple and can be written simply by inspection (figure 2). Thus

$$B_{r+1} = B^2 + B^3 \quad (2.4)$$

$$S_{r+1} = S^2 + BC^2 \quad (2.5)$$

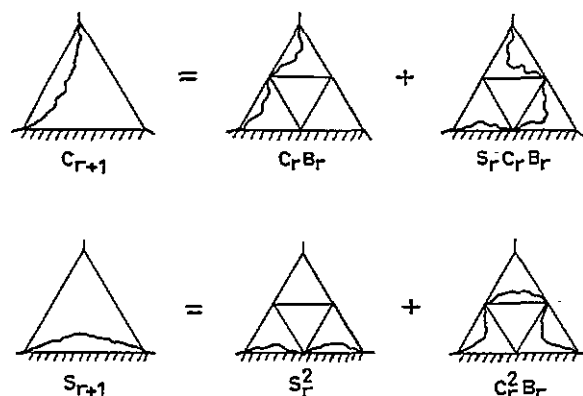


Figure 2. The recursion relations for the two surface functions (C and S) given by equations (2.5)–(2.6) for a Sierpinski-type fractal with $b = 2$.

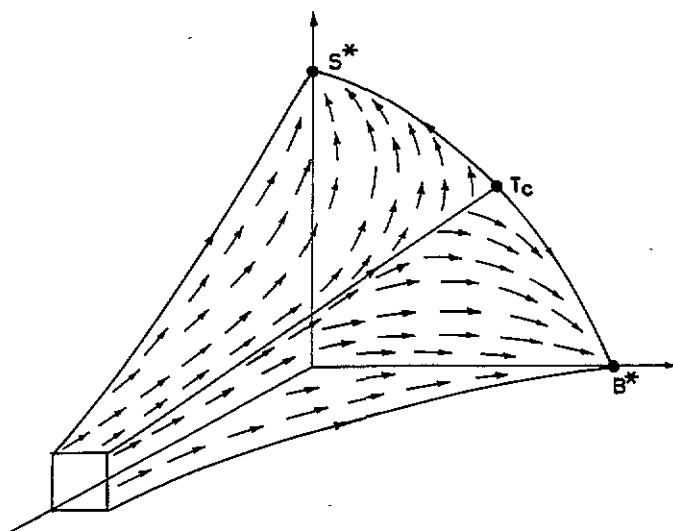


Figure 3. Flow diagram in parameter space for a Sierpinski gasket with rescaling parameter $b=2$. The fixed points discussed in the text are shown.

and

$$C_{r+1} = CB + SCB. \tag{2.6}$$

For notational simplification we drop the suffix r from the right-hand side of recursion relations. Note that (2.4) corresponds to the polymer chain in the bulk and has been studied earlier [20]. It is decoupled from the surface and it remains so under iteration. The starting weight of the functions of (2.4)–(2.6) are chosen to be

$$B_0 = x^2 \tag{2.7}$$

$$S_0 = \omega^2 x^2 \tag{2.8}$$

$$C_0 = t\omega x^2. \tag{2.9}$$

In writing the above equations weights x , ωx and $t\omega x$ were assigned to each visited site in the bulk, surface and layer adjacent to the surface, respectively. More general or complicated initial conditions could be considered by allowing long-range interaction with the wall but it would not change the qualitative behaviour of the phase diagram. Equations (2.4)–(2.6) have been solved and the general flow diagram is given in figure 3. Three non-trivial fixed points are found, whose features are discussed below:

- (i) The fixed point $(B^*, S^*, C^*) = (0.61803, 0, 0)$ corresponds to the bulk state, i.e. disordered state, of the chain with $\nu=0.7986$ as obtained earlier. (For $x = x_c(\omega) = 0.78615$ the fixed point is reached for all $\omega < \omega_c(t)$. Note that $\omega_c(t)$ is a function of t ; for $t=0.5$ the value of $\omega_c(0.5) = 1.1118$.
- (ii) The fixed point $(B^*, S^*, C^*) = (0, 1, 0)$ is reached for all $\omega > \omega_c(t)$ and $x < x_c(\omega)$. This represents the adsorbed state of the polymer chain with $\nu=1$. It should be noted that surface dimension of this fractal family is one and hence the value of ν obtained is consistent with earlier prediction.

(iii) The fixed point $(B^*, S^*, C^*) = (0.61803, 0.61803, 0.61803)$ is obtained for $\omega = \omega_c(t)$. The linearization of equations (2.4)–(2.6) around these fixed points yields two eigenvalues greater than one, i.e.

$$\lambda_s = 1.6709$$

$$\lambda_b = 2.3819.$$

We identify this as a tricritical point. The crossover exponent is found to be

$$\phi = \frac{\ln \lambda_s}{\ln \lambda_b} = 0.5915.$$

At the point of adsorption transition the free energy per monomer is given as [21]

$$f(T) = T \ln x_c \approx (T_c - T)^{2-\alpha} + \text{regular part.}$$

The first term on the RHS of the above equation expresses the leading singular behaviour of the polymer free-energy density. The ‘specific heat’ critical exponent α at the tricritical point is related to ϕ as [21]

$$\alpha = 2 - 1/\phi.$$

For $b=2$, the value of α is found to be 0.3094.

It is straightforward to extend this method to other lattices of this family. However, as the value of b increases the number of possible configurations of different walks increase rapidly. Therefore, the extension of this method to much larger values of b appears difficult as the computer time needed to generate the exact renormalization equation by direct enumeration increases as $\exp(b^2)$. We summarize results for $2 \leq b \leq 6$ in Table 1, in which are listed the fixed point corresponding to the tricritical point of the system, the values of the two eigenvalues which are greater than one and values of the exponents ν , ϕ and α . Note that the values of ν reported here have been obtained earlier by Elezovic *et al* [18]. The value of ϕ is found to decrease as b increases. This can be attributed to the fact that d_F increases with b , making relatively more bonds available in the bulk for the polymer to spread than the surface. At $b=6$ the value of ϕ becomes lower than that of a two-dimensional system and α changes sign.

Numerical analysis of the results for λ_b and λ_s given in Table 1 suggests the following relations between these quantities:

$$\lambda_b = b^{\sqrt{d_F}} \quad (2.9)$$

and

$$\lambda_s = b^{[1+(1-d_F)\nu]/\nu} \quad (2.10)$$

Table 1.

b	d_F	$S^* = B^* = C^*$	λ_b	λ_s	$\nu(b)$	$\phi(b)$	$\alpha(b)$
2	1.5849	0.6180	2.3819	1.6709	0.7986	0.5915	0.3094
3	1.6309	0.5511	3.9919	2.1628	0.7936	0.5573	0.2056
4	1.6609	0.5063	5.8029	2.5418	0.7884	0.5305	0.1151
5	1.6826	0.4745	7.7898	2.8426	0.7840	0.5089	0.0350
6	1.6994	0.4507	9.9360	3.0864	0.7803	0.4908	-0.0374

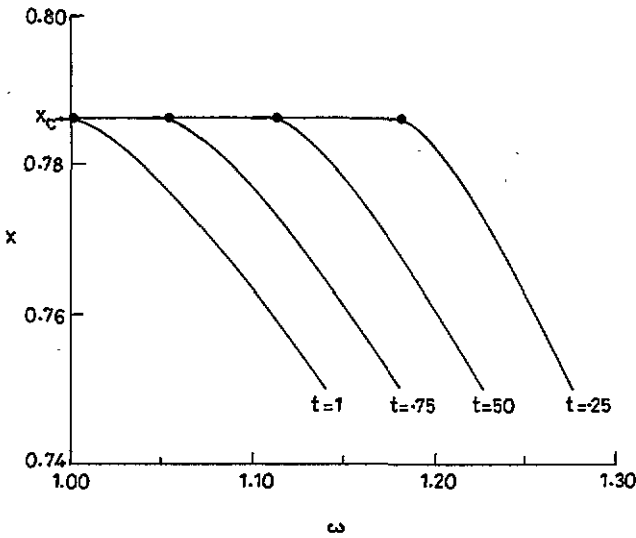


Figure 4. Critical fugacity x as a function of the surface interaction parameter $\omega = \exp(-E_s/k_B T)$ for several values of $t = \exp(-E_t/k_B T)$ for the fractal lattice $b=2$. The tricritical points of adsorption are shown by full circles.

where $\nu = \frac{3}{4}$ is the exactly known correlation length exponent for $b \rightarrow \infty$ (or $d_F = 2$). From (2.9) we find

$$\nu(b) = \ln b / \ln \lambda_b = 1 / \sqrt{d_F} \tag{2.11}$$

This simple looking relation gives values of $\nu(b)$ which are in very good agreement with those found exactly using the RSRG method [18] for $2 \leq b \leq 8$ and the Monte Carlo RG method for $2 \leq b \leq 80$ [24]. For example, the values of $\nu(b)$ given in [18] and [24] for $b=2, 20, 50$ and 80 are, respectively, $0.7986, 0.7560, 0.7399$ and 0.7340 , whereas the values found from equation (2.11) are $0.7943, 0.7485, 0.7396$ and 0.7363 . Equation (2.11) cannot, however, be valid for $b \rightarrow \infty$ as its predictions contradict the results of finite-size scaling [19], a monotonic decrease in the value of $\nu(b)$ as b is increased. From (2.9) and (2.10) we obtain

$$\phi(b) = \ln \lambda_s / \ln \lambda_b = [1 + (1 - d_F)\nu] \sqrt{d_F} / \nu \tag{2.12}$$

This predicts the value of ϕ which lies within the bounds given by (1.1). Equation (2.12) is expected to be valid for moderately large values of b .

In figure 4 the phase diagram in parameters x and ω for a few fixed values of t is plotted. Note that for each t we have a value of $\omega = \omega_c(t)$ such that for $\omega > \omega_c(t)$ an adsorbed phase is found to exist for $x < x_c$. The $\omega < \omega_c(t)$ and $x = x_c$ part of the curve represents the desorbed phase and the point $\omega = \omega_c(t)$ and $x = x_c$ is a tricritical point. As the value of t increases, $\omega_c(t)$ decreases. This is shown in figure 5 in which ω_c is plotted as a function of t for $b=2$. It thus appears that in order to have an unbinding transition either E_s or E_t must be repulsive. If both E_s and E_t are repulsive, polymer will not be adsorbed and similarly a desorbed phase will not be found to exist if both E_s and E_t are attractive. When $E_t=0$ (i.e. $t=1$) the tricritical point is found at $\omega = 1$ ($E_s=0$). These features of the phase diagram exist for all lattices of this family.

3. Determination of ϕ for large b

Consider a large triangle of base b and assume that its one edge (say base) has bond strength E_s and the bond connecting the edge (or surface) to the bulk has interaction zero, i.e. $E_c = 0, t = 1$. As has been shown in the previous section, for this case $E_s = 0$ is the tricritical point. Note that this corresponds to an unperturbed state. Analogous to the scaling ansatz for bulk [19], we assume the following scaling behaviour of the surface:

$$S \approx \frac{1}{b^*} g[\varepsilon b^{1/\nu}, E_s b^{1/\nu_2}] \tag{3.1}$$

where g is a scaling function of two arguments. Here $\exp(\varepsilon) = \exp(\varepsilon^r) = B^r/B^*(\infty)$, where the superscript r stands for order of iteration and $B^*(\infty)$ is the bulk critical fixed point for $b = \infty$. To simplify notation we shall, however, drop the superscript r in the remainder of the article.

By definition $\varepsilon b^{1/\nu}$ and $E_s b^{1/\nu_2}$ are the combination that are unchanged under the renormalization transformation. Thus for an infinite base

$$b \rightarrow b/\Lambda \quad \varepsilon \rightarrow \Lambda^{1/\nu} \quad \text{and} \quad E_s \rightarrow E_s \Lambda^{1/\nu_2}.$$

Equation (3.1) can be rewritten as

$$S \approx \frac{K}{b^*} \exp[f(\varepsilon b^{1/\nu}, E_s b^{1/\nu_2})] \tag{3.2}$$

which defines the function f . Here K is a constant. The exponents x, ν and ν_2 are

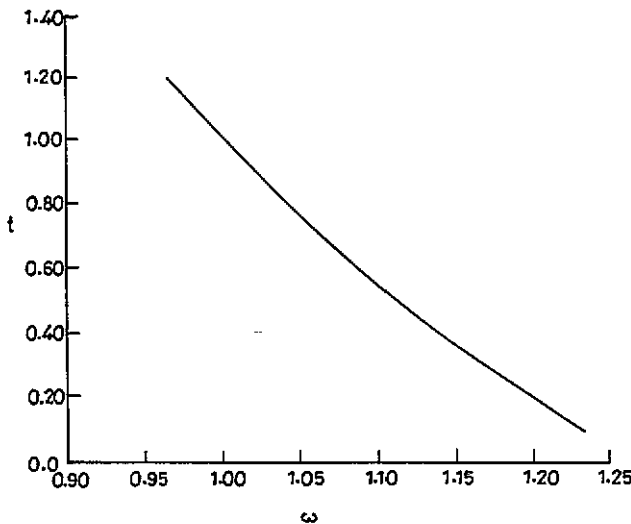


Figure 5. Variation of t as a function of ω_c for $b = 2$.

known from conformal invariance or by coulomb gas methods. Near the fixed point $E_s=0$, we obtain, using Taylor expansion for f

$$f(\epsilon b^{1/\nu}, E_s b^{1/\nu_2}) = f(\epsilon b^{1/\nu}, 0) + E_s b^{1/\nu_2} f'(\epsilon b^{1/\nu}, 0) + \dots \tag{3.3}$$

Now at large b , one has

$$\lambda_s \approx \left. \frac{dS'}{dS} \right|_{S^*=C^*=B^*}$$

where $S' = S_{r+1}$, $S = S_r$ and λ_s is the largest eigenvalue of

$$\begin{vmatrix} \frac{\partial S'}{\partial S} & \frac{\partial S'}{\partial C} \\ \frac{\partial C'}{\partial S} & \frac{\partial C'}{\partial C} \end{vmatrix}_{S^*=C^*=B^*}$$

This is in accordance with the results of the previous section where, even for small values of b , the off-diagonal elements were found to be small.

The average number of contacts of the random self-avoiding walk with the surface is given as

$$\left. \frac{S}{S'} \frac{dS'}{dS} \right|_{S^*=C^*=B^*}$$

at $E_s=0$. Now using the definition of S and f we find

$$\lambda_s = b^{1/\nu_2} f'(\epsilon b^{1/\nu}, 0). \tag{3.4}$$

For self-avoiding walks at large b Dhar [19] has shown that $\epsilon^* b^{1/\nu} \approx (a \ln b)^{1/2\nu}$ and that there are roughly $b^{1/\nu} (\ln b)^{2\nu-1/2\nu}$, number of steps. As these are dense SAWs, they fill the plane with roughly uniform density. So the number of contacts on the edge is

$$\text{average density} \times \text{edge length} \approx k_2 b^{1/\nu+1-d_F} (\ln b)^{2\nu-1/2\nu} \tag{3.5}$$

where k_2 is a constant, which implies that

$$\lambda_s \approx k_2 b^{1/\nu+1-d_F} (\ln b)^{2\nu-1/2\nu}. \tag{3.6}$$

This leads to the following expression for $\phi(b)$

$$\phi(b) = \frac{\ln \lambda_s}{\ln \lambda_b} = 1 + \nu(1-d_F) \left[1 - \frac{2\nu-1}{2} \frac{\ln \ln b}{\ln b} + \text{terms of order } 1/\ln b \right]. \tag{3.7}$$

From this equation we note that for $b \rightarrow \infty$

$$\phi = 1 + \nu(1-d_F) = 0.25 \tag{3.8}$$

where $\nu = \frac{3}{4}$. Since $\ln \ln b / \ln b \approx (2 - \bar{d}(b))$, the first correction to ϕ term due to finite b is proportional to $(2 - \bar{d}(b))$. A similar relation has also been found for bulk $\nu(b)$ and $\gamma(b)$ [19].

Since $(2\nu-1)/2$ is positive for $\nu = \frac{3}{4}$, the first correction term in (3.7) is negative which, when multiplied by $(1-d_F)\nu$, makes a positive contribution to $\phi(b)$. This implies that $\phi(b)$ approaches the $\frac{1}{4}$ value monotonically as b is increased. In figure 6 $\phi(b)$ is plotted as a function of $1/\ln b$. We do not plot $\phi(b)$ against $\ln \ln / \ln b$, as this

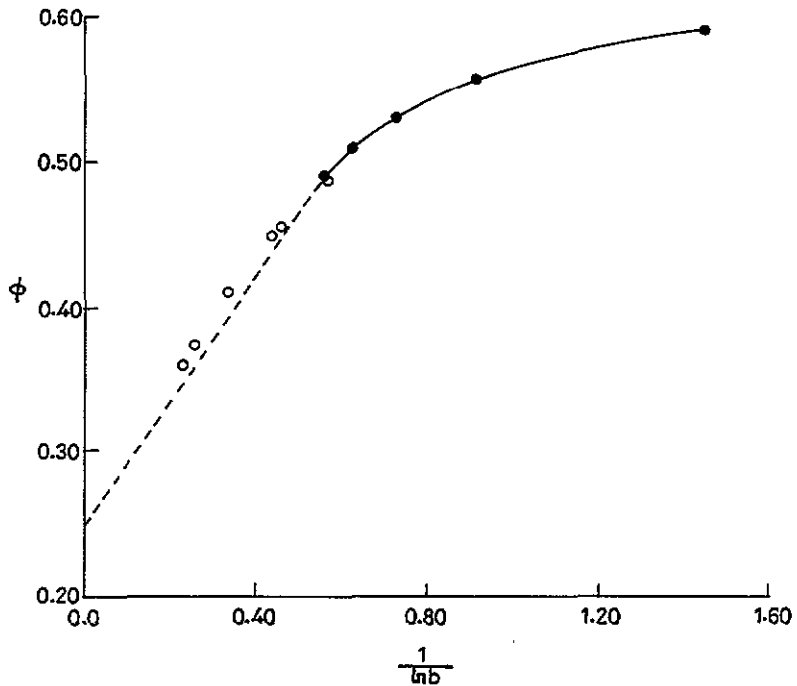


Figure 6. Variation of crossover exponent ϕ as a function of $(\ln b)^{-1}$. Full circles denote the exact values given in Table 1 and the dashed line represents the interpolated region. We have used $\phi(b)=0.25$ at $b=\infty$ in the interpolation. Open circles denote the values obtained by equation (2.12).

involves an additional free parameter. In this plot we have taken $\phi(b)$ for $2 \leq b \leq 6$ the exact values given in Table 1 (shown in the figure by full circles) and for $b = \infty$ the value given by (3.8). The dashed line shows the interpolated region obtained by using spline fitting with tension factor two using software GRAPHIER (Version 1.75, 1988 Developed by Golden Software, Inc.). The values of $\phi(b)$ obtained by (2.12) are also shown in the figure by open circles. Equation (2.12) is seen to yield reasonable values of $\phi(b)$ for a wide range of b .

Implicit in the analysis given above is that the polymer density at the tricritical point is isotropic. This is justified from the fact that the proximal exponent m found from (1.2) using the values of ϕ and ν given in Table 1 is nearly equal to zero particularly for large b . We may also note that the transverse correlation length $\xi_{\perp}(x_c)$ associated to C , and the parallel one $\xi_{\parallel}(x_c)$ associated to S are equal confirming the isotropic distribution of the polymer chain.

4. Discussion

The problem of adsorption of a polymer chain on a boundary of a fractal lattice considered here differs from that of a regular lattice. In the latter case, there is an entropy loss per monomer on the surface because it is impenetrable: once a monomer lies on the surface, the following segment has a restricted orientational choice. This introduces anisotropy in the distribution of monomers and in the correlation lengths

corresponding to parallel and perpendicular distributions. In the case of a fractal, on the other hand, the impenetrable surfaces exist in the bulk lattice because of the construction and therefore, the presence of interaction on a surface does not entail much additional loss of conformational entropy to the polymer. This situation may be compared to the one existing in the case of a penetrable surface of a regular lattice in which both correlation lengths (parallel and perpendicular to the surface) are equal. Enumeration suggests that in the case of penetrable surface $\phi \approx 1 - \nu$ [12] which is in agreement with the result reported above for $b \rightarrow \infty$. Thus, in spite of the fact that even for $b \rightarrow \infty$, when both the spectral and Hausdorff dimension of the lattice tend to 2, the critical exponent ϕ is not equal to the two-dimensional value $\phi = \frac{1}{2}$ should not be a surprise.

In the finite size scaling theory given in section 3, one is concerned with the change in the renormalization equation as scale factor is changed. This differs from the usual case, where the renormalization equations are unchanged as the lengths are rescaled but the coupling constants are changed so that correlation functions of a Hamiltonian at a length scale L , are related to those of the transformed Hamiltonian at length scale L/b . In the treatment given in section 3 one determines how the recursion equations for the b -fractal are related to those of (say) $2b$ -fractal. This analysis clearly indicates that the relation $\phi = 1 - \nu$ is satisfied only for a lattice for which the rescaling parameter b approaches infinity. For finite b , corrections expressed in powers of $\bar{\epsilon} = 2 - \bar{d}(b)$ (which measures the difference of the spectral dimension of the lattice from 2) arise.

Acknowledgment

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Appendix 1

Here we report the recursion relations for $b=3, 4, 5$ and 6 of 2D Sierpinski type fractals.

$$b=3$$

$$B_{r+1} = B^3 + 3B^4 + B^5 + 2B^6$$

$$S_{r+1} = S^3 + 2SBC^2 + 2SC^2B^3 + C^2B^2 + C^2B^3$$

$$C_{r+1} = 2SCB^2 + S^2CB^2 + S^2CB^3 + CB^2 + CB^3 + C^3B^3$$

$$b=4$$

$$B_{r+1} = B^4 + 6B^5 + 6B^6 + 9B^7 + 9B^8 + 9B^9 + 4B^{10}$$

$$S_{r+1} = c^2[B^3 + 3B^4 + B^5 + 2B^6] + C^4[B^2 + 2B^4 + 2B^6]$$

$$+ S[C^2(2B^2 + 2B^3 + 4B^4 + 4B^5 + 2B^6)]$$

$$+ S^2[C^2(3B + 4B^3 + B^4 + 7B^5 + 2B^6)] + S^4$$

$$\begin{aligned}
C_{r+1} = & C[B^3 + 3B^4 + B^5 + 2B^6] + C^3[3B^4 + 2B^5 + 3B^6] \\
& + S[C(3B^3 + 2B^4 + B^5 + B^6) + C^3(2B^4 + 4B^5 + 2B^6)] \\
& + S^2C[3B^3 + 2B^4 + B^5 + B^6] \\
& + S^3C[B^3 + 3B^4 + B^5 + 2B^6]
\end{aligned}$$

$$b = 5$$

$$\begin{aligned}
B_{r+1} = & B^5 + 10B^6 + 20B^7 + 30B^8 + 54B^9 + 68B^{10} + 98B^{11} + 94B^{12} + 86B^{13} + 38B^{14} \\
& + 16B^{15}
\end{aligned}$$

$$\begin{aligned}
S_{r+1} = & C^2[B^4 + 6B^5 + 6B^6 + 9B^7 + 9B^8 + 9B^9 + 4B^{10}] \\
& + C^4[2B^3 + 2B^4 + 6B^5 + 6B^6 + 10B^7 + 14B^8 + 14B^9 + 6B^{10}] \\
& + S[C^2(2B^3 + 6B^4 + 8B^5 + 20B^6 + 14B^7 + 24B^8 + 8B^9 + 6B^{10})] \\
& + C^4(3B^2 + 8B^4 + 2B^5 + 17B^6 + 8B^7 + 32B^8 + 12B^9 + 8B^{10}) \\
& + S^2C^2[3B^2 + 3B^3 + 8B^4 + 9B^5 + 23B^6 + 26B^7 + 26B^8 + 12B^9 + 6B^{10}] \\
& + S^3C^2[4B + 6B^3 + 2B^4 + 14B^5 + 12B^6 + 34B^7 + 22B^8 + 14B^9 + 8B^{10}] \\
& + S^5
\end{aligned}$$

$$\begin{aligned}
C_{r+1} = & C[B^4 + 6B^5 + 6B^6 + 9B^7 + 9B^8 + 9B^9 + 4B^{10}] \\
& + C^3[6B^5 + 9B^6 + 18B^7 + 20B^8 + 22B^9 + 7B^{10}] \\
& + C^5[B^6 + 3B^7 + 3B^8 + 6B^9 + 2B^{10}] \\
& + S[C(4B^4 + 8B^5 + 5B^6 + 11B^7 + 6B^8 + 9B^9 + B^{10})] \\
& + C^3(8B^5 + 15B^6 + 26B^7 + 35B^8 + 26B^9 + 7B^{10}) \\
& + S^2[C(6B^4 + 6B^5 + 8B^6 + 6B^7 + 6B^8 + 4B^9 + 4B^{10})] \\
& + C^3(3B^5 + 15B^6 + 14B^7 + 28B^8 + 15B^9 + 10B^{10}) \\
& + S^3C[4B^4 + 8B^5 + 56B^6 + 11B^7 + 6B^8 + 9B^9 + B^{10}] \\
& + S^4C[B^4 + 6B^5 + 6B^6 + 9B^7 + 9B^8 + 9B^9 + 4B^{10}]
\end{aligned}$$

$$b = 6$$

$$\begin{aligned}
B_{r+1} = & B^6 + 15B^7 + 50B^8 + 90B^9 + 201B^{10} + 327B^{11} + 604B^{12} + 898B^{13} + 1392B^{14} \\
& + 1687B^{15} + 1985B^{16} + 1720B^{17} + 1371B^{18} + 757B^{19} + 334B^{20} + 68B^{21}
\end{aligned}$$

$$\begin{aligned}
S_{r+1} = & C^2(B^5 + 10B^6 + 20B^7 + 30B^8 + 54B^9 + 68B^{10} + 98B^{11} + 94B^{12} + 86B^{13} + 38B^{14} \\
& + 16B^{15}) \\
& + C^4(3B^4 + 8B^5 + 15B^6 + 34B^7 + 42B^8 + 102B^9 + 127B^{10} + 208B^{11} \\
& + 206B^{12} + 196B^{13} + 90B^{14} + 30B^{15}) \\
& + C^6(B^3 + 4B^5 + B^6 + 10B^7 + 6B^8 + 27B^9 + 25B^{10} + 59B^{11} + 40B^{12} \\
& + 48B^{13} + 22B^{14} + 8B^{15}) \\
& + S[C^2(2B^4 + 12B^5 + 20B^6 + 58B^7 + 74B^8 + 138B^9 + 154B^{10} + 212B^{11}
\end{aligned}$$

$$\begin{aligned}
 &+ 154B^{12} + 134B^{13} + 54B^{14} + 18B^{15}) \\
 &+ C^4(6B^3 + 6B^4 + 24B^5 + 28B^6 + 76B^7 + 100B^8 + 224B^9 + 304B^{10} \\
 &+ 444B^{11} + 406B^{12} + 344B^{13} + 156B^{14} + 36B^{15})] \\
 &+ S^2[C^2(3B^3 + 9B^4 + 15B^5 + 39B^6 + 62B^7 + 145B^8 + 171B^9 + 271B^{10} \\
 &+ 228B^{11} + 228B^{12} + 136B^{13} + 74B^{14} + 22B^{15}) \\
 &+ C^4(6B^2 + 18B^4 + 6B^5 + 50B^6 + 40B^7 + 148B^8 + 164B^9 + 396B^{10} \\
 &+ 346B^{11} + 412B^{12} + 260B^{13} + 146B^{14} + 36B^{15})] \\
 &+ S^3C^2(4B^2 + 4B^3 + 12B^4 + 14B^5 + 44B^6 + 62B^7 + 154B^8 + 214B^9 \\
 &+ 292B^{10} + 274B^{11} + 232B^{12} + 164B^{13} + 82B^{14} + 24B^{15}) \\
 &+ S^4C^2(5B + 8B^3 + 3B^4 + 21B^5 + 22B^6 + 69B^7 + 96B^8 + 224B^9 + 240B^{10} \\
 &+ 275B^{11} + 229B^{12} + 159B^{13} + 106B^{14} + 24B^{15}) + S^6
 \end{aligned}$$

and

$$\begin{aligned}
 C_{r+1} = &C(B^5 + 10B^6 + 20B^7 + 30B^8 + 54B^9 + 68B^{10} + 98B^{11} + 94B^{12} + 86B^{13} + 38B^{14} \\
 &+ 16B^{15}) \\
 &+ C^3(10B^6 + 28B^7 + 58B^8 + 113B^9 + 185B^{10} + 264B^{11} + 298B^{12} + 239B^{13} \\
 &+ 115B^{14} + 37B^{15}) \\
 &+ C^5(5B^7 + 16B^8 + 36B^9 + 76B^{10} + 110B^{11} + 148B^{12} + 129B^{13} + 74B^{14} \\
 &+ 17B^{15}) \\
 &+ S[C(5B^5 + 20B^6 + 21B^7 + 47B^8 + 53B^9 + 89B^{10} + 86B^{11} \\
 &+ 103B^{12} + 57B^{13} + 27B^{14} + 7B^{15}) \\
 &+ C^3(20B^6 + 44B^7 + 116B^8 + 198B^9 + 356B^{10} + 448B^{11} \\
 &+ 514B^{12} + 314B^{13} + 172B^{14} + 35B^{15}) \\
 &+ C^5(3B^7 + 22B^8 + 34B^9 + 81B^{10} + 108B^{11} + 165B^{12} + 128B^{13} + 86B^{14} \\
 &+ 14B^{15})] \\
 &+ S^2[C(10B^5 + 19B^6 + 28B^7 + 40B^8 + 51B^9 + 63B^{10} + 71B^{11} + 70B^{12} \\
 &+ 59B^{13} + 24B^{14} + 11B^{15}) \\
 &+ C^3(15B^6 + 57B^7 + 99B^8 + 242B^9 + 325B^{10} + 511B^{11} + 458B^{12} \\
 &+ 370B^{13} + 146B^{14} + 48B^{15})] \\
 &+ S^3[C[10B^5 + 19B^6 + 28B^7 + 40B^8 + 51B^9 + 63B^{10} + 71B^{11} + 70B^{12} \\
 &+ 59B^{13} + 24B^{14} + 11B^{15}) \\
 &+ C^3(4B^6 + 36B^7 + 66B^8 + 127B^9 + 204B^{10} + 286B^{11} + 312B^{12} \\
 &+ 250B^{13} + 138B^{14} + 38B^{15})] \\
 &+ S^4C(5B^5 + 20B^6 + 21B^7 + 47B^8 + 53B^9 + 89B^{10} + 86B^{11} + 103B^{12}
 \end{aligned}$$

$$\begin{aligned}
&+ 57B^{13} + 27B^{14} + 7B^{15}) \\
&+ S^5 C (B^5 + 10B^6 + 20B^7 + 30B^8 + 54B^9 + 68B^{10} \\
&+ 98B^{11} + 94B^{12} \\
&+ 86B^{13} + 38B^{14} + 16B^{15})
\end{aligned}$$

where for notational simplification the subscript r has been dropped from the right-hand side.

References

- [1] de Gennes P G 1987 *Adv. Colloid Interface Sci.* **27** 189
- [2] Napper D 1983 *Polymeric Stabilization of Colloidal Dispersions* (New York: Academic)
- [3] Hammerley J M, Torrie G M and Whittington S G 1982 *J. Phys. A: Math. Gen.* **15** 539
- [4] Bouchaud E, Auvray L, Cotton J P, Daoud M, Farnoux B and Janninko G 1988 *Prog. Surf. Sci.* **27** 5
- [5] Diehl H W and Dietrich S 1981 *Phys. Rev. B* **24** 2878
- [6] Eisenriegler E, Kremer K and Binder K 1982 *J. Chem. Phys.* **77** 3171, 6296
- [7] de Gennes P G 1979 *Scaling Concepts in Polymer Physics* (Ithaca: Cornell University)
- [8] Binder K 1983 *Phase Transition and Critical Phenomena* ed C Domb and J L Lebowitz vol 7 (New York: Academic)
- [9] de Gennes P G 1976 *J. Physique* **37** 1445
- [10] Kremer K 1983 *J. Phys. A: Math. Gen.* **16**, 4333
- [11] Bouchaud E and Vannimenus J 1989 *J. Physique* **50** 2931
- [12] Ischinabe T 1982 *J. Chem. Phys.* **77** 6296; 1984 *J. Chem. Phys.* **80** 1318
- [13] Duplantier B and Saleur H 1987 *Phys. Rev. Lett.* **59** 539
- [14] Burkhardt T W, Eisenriegler E and Guim I 1989 *Nucl. Phys. B* **316** 559
Guim I and Burkhardt T W 1989 *J. Phys. A: Math. Gen.* **22** 1131
- [15] de Gennes P G and Pincus P 1983 *J. Physique Lett.* **44** L-241
- [16] Orlandini E, Seno F, Stella A L and Tesi M C 1992 *Phys. Rev. Lett.* **68** 488
- [17] Hilfer R and Bluemen A 1984 *J. Phys. A: Math Gen.* **17** L537
- [18] Elezovic S, Knezevic M and Milosevic S 1987 *J. Phys. A: Math Gen.* **20** 1215
- [19] Dhar D 1988 *J. Physique* **49** 397
- [20] Dhar D 1978 *J. Math. Phys.* **19** 5
- [21] Derrida B and Herrmann H 1984 *J. Physique* **44** 1365
- [22] Milosevic S and Zivic I 1991 *J. Phys. A: Math. Gen.* **24** L833